Simple and exactly solvable model for queue dynamics

Yūki Sugiyama
Division of Mathematical Science, City College of Mie, Tsu, Mie 514-01, Japan

Hiroyasu Yamada*
Department of Physics, Nagoya University, Nagoya 464-01, Japan

(Received 17 June 1996)

We present a model for queue dynamics that is very simple but provides the essential property for such dynamics. The model has the traveling cluster solution, which is derived analytically, as well as the homogeneous flow solution. The cluster solution is a simple example of a pattern formation in diffusion system, which is seen in the phenomena of traffic jams and the slugging of granular flow. [S1063-651X(97)07205-X]

PACS number(s): 05.60.+w, 05.70.Fh, 46.10.+z, 89.40.+k

The concept of queue dynamics offers simple one-dimensional models for socioeconomic and complex multi-body physical systems, such as the problems of traffic and granular flow, pedestrians, gases, and fluids [1–8]. We propose a simple and exactly solvable model of this kind. Our model describes the general aspects of queue dynamics and can be widely applicable to those problems. But the following discussion is presented with the terminology of traffic problems.

The model is

$$\ddot{x}_n = a\{V(\Delta x_n) - \dot{x}_n\}, \tag{1}$$

where

$$\Delta x_n = x_{n-1} - x_n \tag{2}$$

for each car number n ($n=1,2,\ldots$). x_n is the position of the nth car, Δx_n is the headway of that car. The overdot denotes the time derivative. a is a sensitivity constant, which we set the same value for all drivers. The function $V(\Delta x_n)$, which is called the optimal velocity (OV) function, is

$$V(\Delta x_n) = v_{\max}\theta(\Delta x_n - d), \tag{3}$$

where θ is the Heaviside function. It decides the optimal velocity (the safety velocity) according to the headway: (a) if the headway is less than d, a car should stop; (b) if the headway is larger than d, a car can accelerate to move with the maximum velocity $v_{\max}$.

For both cases, the movement of each car is easily derived as follows. (a) In the case of $\Delta x_n < d$,

$$x_n(t) = x_n(t_0) + \frac{\dot{x}_n(t_0)}{a}\{1 - e^{-a(t-t_0)}\}, \tag{4}$$

with the initial condition $t=t_0, x(t_0), \dot{x}(t_0)$. (b) In the case of $\Delta x_n \geqslant d$,

$$x_n(t) = x_n(t_0) + v_{\max}(t-t_0) - \frac{v_{\max} - \dot{x}_n(t_0)}{a}\{1 - e^{-a(t-t_0)}\}, \tag{5}$$

with the same initial condition.

It is easy to understand that the system has stable flows when the headway is far from d. If the headway of all cars is less than d, all cars stop. When all cars move with the same velocity $v_{\max}$ and with the headway larger than d, this homogeneous flow is also stable. The latter can be considered as ‘‘free-driving flow’’ with no jam.

We are interested in obtaining the solution of the jam's flow. For this purpose, we should consider the two basic processes of a car moving. Suppose a jam exists in a lane. A car moves from the ‘‘free-driving region’’ into the jam and another car escapes from the jam to the free-driving region. These two processes can be expressed by the above two solutions with appropriate connection conditions. We assume the ideal case: Cars stop in a jam with the same distance Δx_J ($<d$) and move at the maximum velocity $v_{\max}$ with the same distance Δx_F ($>d$) in the free-driving region.

First, we investigate the process of a car moving from a jam to a free-driving region. When the headway of the front car in a cluster is d, we set $t=t_0$. At this time the position of the car is set as $x_0=0$. The nth car in the jam moves as the formulas

$$x_n(t) = -n\Delta x_J \quad (t_0 \leqslant t < t_n), \tag{6}$$

$$x_n(t) = -n\Delta x_J + v_{\max}\left\{(t-t_n) - \frac{1}{a}[1 - e^{-a(t-t_n)}]\right\} \quad (t_n \leqslant t), \tag{7}$$

where t_n is the time when the headway of the nth car is d. After t_n the car begins to move and escapes from jam. We note that $t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots$ and t_n is defined by $\Delta x_n(t_n) = d$. This equation is written from Eqs. (6) and (7) as

$$\Delta x_J + v_{\max}\left\{(t_n - t_{n-1}) - \frac{1}{a}[1 - e^{-a(t_n - t_{n-1})}]\right\} = d. \tag{8}$$

*Present address: Bio-Mimetic Control Research Center (RIKEN), Nagoya 463, Japan.

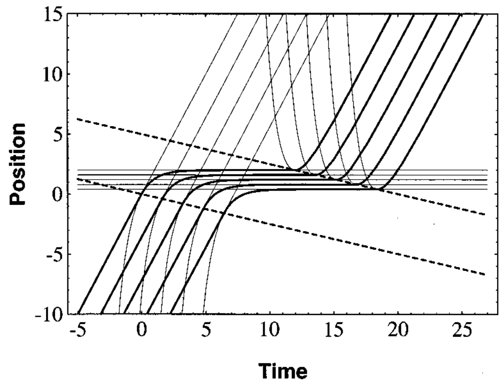

FIG. 1. Example of jam flow. The movements of five successive cars are shown by bold lines, which form a jam cluster. The dashed lines show the movement of a jam cluster.

We have derived the sequence of equations for the definition of t_n ($n=1,2, \dots$), which has the solution for $t_n > t_{n-1}$. It is easily obtained by setting $\tau = t_1 - t_0 = \cdots = t_n - t_{n-1} = \cdots$ (>0); this is a unique solution, which is given by

$$\Delta x_J + v_{\max}\left\{\tau - \frac{1}{a}(1-e^{-a\tau})\right\} = d. \tag{9}$$

It is easily seen that the velocity of the nth car, $\dot{x}_n(t)$ of Eq. (7), converges to $v_{\max}$ for sufficient time ($t \gg t_n$). This means that the car reaches the free-driving region and the headway of this car becomes Δx_F, which is expressed as $\lim_{t\to\infty} \Delta x_n(t) = \Delta x_F$, using Eq. (7). Thus we have obtained the simple relation

$$\Delta x_J + v_{\max}\tau = \Delta x_F. \tag{10}$$

Next, we make an analogous investigation for the process of a car moving from the free-driving region into the jam. When the headway of the car positioned just behind a cluster is d, we set $t=t_0$ and the position of the car $x_0=0$. The nth car in the free-driving region moves as the formulas.

$$x_n(t) = -n\Delta x_F + v_{\max}(t-t_0) \quad (t_0 \leq t < t_n), \tag{11}$$

$$x_n(t) = -n\Delta x_F + v_{\max}\left\{(t_n - t_0) + \frac{1}{a}[1-e^{-a(t-t_n)}]\right\}$$

$$(t_n \leq t), \tag{12}$$

where t_n is the time when the headway of the nth car is d. After t_n the car begins to decelerate and moves into jam. We note that $t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots$ and t_n is defined by $\Delta x_n(t_n) = d$. As in the previous case, we can solve this equation from Eqs. (11) and (12) with the conditions $\tau = t_1 - t_0 = \cdots = t_n - t_{n-1} = \cdots$ (>0) as

$$\Delta x_F + v_{\max}\left\{-\tau + \frac{1}{a}(1-e^{-a\tau})\right\} = d. \tag{13}$$

The velocity of the nth car, $\dot{x}_n(t)$ of Eq. (12), converges to 0 for sufficient time ($t \gg t_n$), which means that the car reaches a cluster and stops. So the headway of this car becomes Δx_J, which is expressed as $\lim_{t\to\infty}\Delta x_n(t) = \Delta x_J$ using Eq. (12). Again, we have obtained the same relation as Eq. (10), which means that the value of τ is the same for both processes we have discussed above.

Now we can solve Eqs. (9), (10), and (13). Thus Δx_F, Δx_J, and τ can be expressed using a, d, and $v_{\max}$ as

$$\Delta x_F = d + \frac{v_{\max}\tau}{2}, \quad \Delta x_J = d - \frac{v_{\max}\tau}{2}. \tag{14}$$

τ is determined by the equation

$$a\tau = 2(1-e^{-a\tau}), \tag{15}$$

which has the solution $a\tau \simeq 1.59$. As a result, each car is moving in the same manner as the car moving in front with the time delay τ, which is proportional to the inverse of sensitivity $1/a$. The collective movement of these cars forms a cluster.

In Fig. 1 the movement of several successive cars is shown. Their orbits consist of Eqs. (11), (12), (6), and (7) with the time delay τ. Figure 2 shows the velocity of the cars with time development, which is easily obtained as the time derivatives of these formulas. Specifically, it takes an infinite

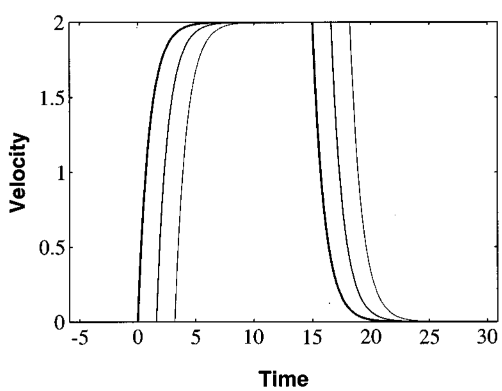

FIG. 2. Changing of car velocity of three successive cars. The maximum velocity is set as $v_{\max}=2$. We can see a kinklike shape.

TABLE I. Movement of two cars from a jam to a free-driving region.

t	x_{n-1}	x_n
$t_0 < t < t_{n-1}$	$\Delta x_{n-1} < d$ [a]	$\Delta x_n < d$ [a]
$t_{n-1} < t < t_n$	$\Delta x_{n-1} > d$ [b]	$\Delta x_n < d$ [a]
$t_n < t$	$\Delta x_{n-1} > d$ [b]	$\Delta x_n > d$ [b]

[a]For Eq. (6).
[b]For Eq. (7).

TABLE II. Movement of two cars from a free-driving region into a jam.

t	x_{n-1}	x_n
$t_0 < t < t_{n-1}$	$\Delta x_{n-1} > d$ [a]	$\Delta x_n > d$ [a]
$t_{n-1} < t < t_n$	$\Delta x_{n-1} < d$ [b]	$\Delta x_n > d$ [a]
$t_n < t$	$\Delta x_{n-1} < d$ [b]	$\Delta x_n < d$ [b]

[a]For Eq. (11).
[b]For Eq. (12).

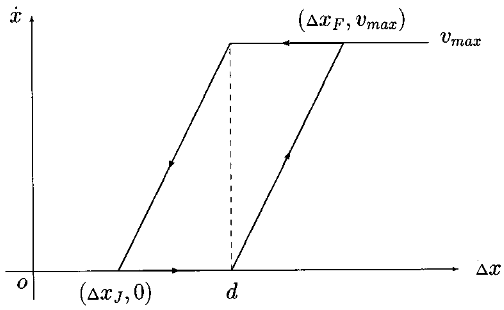

FIG. 3. Hysteresis loop of jam flow together with the OV function $V(\Delta x)$. Each car moves along this loop in the direction of the arrow with time development.

time for the car's velocity to reach $v_{\max}$ or 0. Practically, an appropriate finite time is enough to be considered as infinite, as can be seen in Fig. 2. In Fig. 1 a cluster moves backward against the direction of the moving car . The velocity of the cluster is defined by the moving of the front position of the cluster, which is obtained by Eq. (6) or (7) as

$$v_{\text{jam}}=\frac{x_n(t_n)-x_{n-1}(t_{n-1})}{t_n-t_{n-1}}=-\frac{\Delta x_J}{\tau}. \tag{16}$$

The same result is given by the moving of the rear point of the cluster from Eq. (11) or Eq. (12) with Eq. (10).

The profile of a jam's flow is clearly described as the trajectory in the phase space of headway and velocity $(\Delta x,\dot{x})$. In order to draw this, it is enough to check the movement of two successive cars.

First, we check the process of a car moving from a jam to a free-driving region. The moving of the $(n-1)$th and nth cars is divided into three stages presented in Table I. According to this table, we can obtain the relation between Δx_n and $\dot{x}_n$ using Eqs. (14) and (15). (i) $t_0 \leqslant t < t_{n-1}$: The headway of the nth car is Δx_J and its velocity is 0. The nth car stays at point $(\Delta x_J,0)$, which means that the car stays in a jam. (ii) $t_{n-1} \leqslant t < t_n$: The velocity $\dot{x}_n$ is 0. The headway changes $\Delta x_J \leqslant \Delta x_n < d$. The trajectory is the line $(\Delta x_J,0)$–$(d,0)$. (iii) $t_n \leqslant t < \infty$: In this period, we derive the relation $\Delta x_n = d+\tau\dot{x}_n/2 (0 \leqslant \dot{x}_n < v_{\max})$. The trajectory is the line $(d,0)$–$(\Delta x_F, v_{\max})$.

Next, we turn to the car moving from a free-driving region into a jam. The process is also divided into three stages in Table II. (i) $t_0 \leqslant t < t_{n-1}$: The headway of the nth car is Δx_F and its velocity is $v_{\max}$. The nth car stays at the point $(\Delta x_F, v_{\max})$, which means that the car moves in a free-driving region. (ii) $t_{n-1} \leqslant t < t_n$: The velocity $\dot{x}_n$ is $v_{\max}$. The headway changes $\Delta x_F \geqslant \Delta x_n > d$. Thus the trajectory is the line $(\Delta x_F, v_{\max})$–$(d, v_{\max})$. (iii) $t_n \leqslant t < \infty$: In this period we derive the relation $\Delta x_n = \Delta x_J + \tau\dot{x}_n/2$ $(v_{\max} \geqslant \dot{x}_n > 0)$. The trajectory is the line $(d, v_{\max})$–$(\Delta x_J,0)$.

We summarize the above results in Fig. 3. The car movement of the jam's flow solution is represented as the square-shaped closed loop in phase space. All cars are moving along this loop in a stable way, which can be understood as a limit cycle. A similar profile to this model was seen in our previous work (the OV model), whose OV function is the hyperbolic tangent instead of Eq. (3). We called that a "hysteresis loop," which was found by numerical simulation. The loop is attractive in phase space [4].

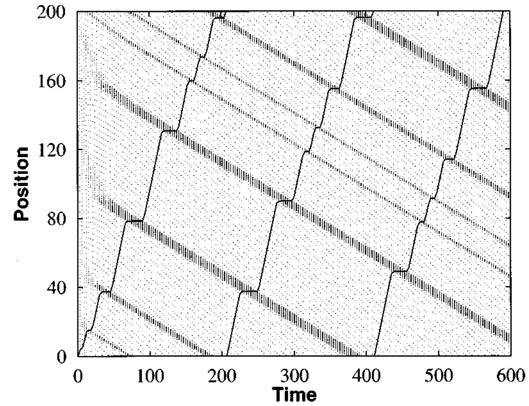

FIG. 4. Simulation data of the positions of all cars in the circuit with time development. The solid line shows the orbit of a sample car.

The size of the hysteresis loop $\Delta x_F - \Delta x_J = v_{\max}\tau$, which determines the amplitude of the cluster, is characterized by the induced time delay τ. It is proportional to the inverse of the sensitivity $1/a$. The loop shrinks to the vertical line $\Delta x = d$ $(0 < \dot{x} \leqslant v_{\max})$ as $a \to \infty$, in which case the jam does not appear. We should note that all the above results for the cluster solution do not depend on the total number of cars and the length of the lane. We do not need to set periodic boundary conditions on the lane.

Finally, we check the above results against the simulation data. The aim is to know how well the analytic solution is realized, which we have obtained by the infinite-time approximation. We set the parameters of the model as $a=1$, $d=2$, and $v_{\max}=2$. In this case $\Delta x_F \simeq 3.59$ and $\Delta x_J \simeq 0.41$ are derived from Eq. (14). The simulation is performed by putting cars on the circuit whose length $L=200$ and the total number of cars $N=100$.

Figure 4 is a plot of the position of all cars with time development. The initial condition is set as all cars are uniformly distributed with the distance $\Delta x = L/N = 2$ $(=d)$ and move with the same velocity $\dot{x}=1$. In this case, the initial movement is highly unstable. We can observe the growth of clusters, which are stably moving backward with the same velocity, a value that is in agreement with the analytic result (16).

Figure 5 is a snapshot of the headway distribution of the jam's flow on the circuit. In this case cars are gathered in one big stable cluster, whose simulation is performed with a different initial condition from the previous case. In both cases, the values of the headway in the jam and the free-driving region are the same as the analytic results. The total number of cars in jams is denoted by N_J, which is given by the following formula, which does not depend on the initial conditions:

$$(N-N_J)\Delta x_F + N_J \Delta x_J \simeq L. \tag{17}$$

The numerical result of N_J is the same as that of the analytic prediction.

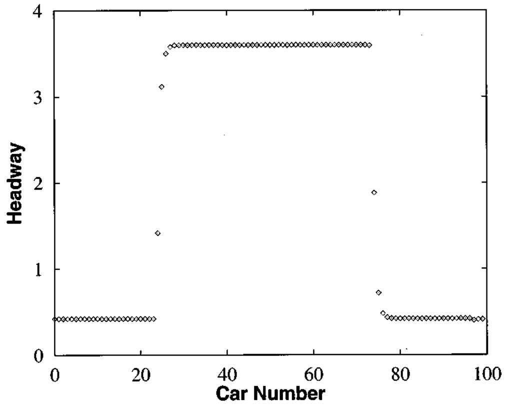

FIG. 5. Simulation data of the distribution of headway of each car. The data agree with the analytic results: $\Delta x_J \simeq 0.41$ and $\Delta x_F \simeq 3.59$.

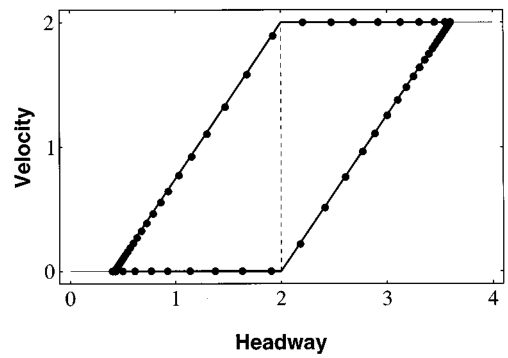

FIG. 6. Simulation data of the car points of jam flow in the phase space together with the hysteresis loop (Fig. 3) and the OV function.

Figure 6 is a plot of car points in phase space accumulated over the steps after the cluster becomes stable. The data points are on the hysteresis loop, which is derived analytically. We note that the hysteresis loop is the same for each cluster in both simulations. Actually, this profile does not depend on the initial condition nor on the car density N/L. This fact is understood by the derivation of analytic results. In conclusion, all simulation data show that the jam's flow is realized just as the analytic solution predicts.

We should discuss the stability of the jam's flow solution and of the homogeneous flow solution. The stability of the homogeneous flow is guaranteed by the linear analysis [4]. It is valid in the case that the averaged headway (the inverse of the car density) is far from d. However, the value is close to d, the homogeneous flow is unstable, and the jam's flow is organized with time development, which is observed in Fig. 4, as an example. The car density is the control parameter even in our simple model and the cluster appears when the density exceeds some critical value.

To summarize, we have proposed a simple model for queue dynamics, in which the traveling cluster solution (the jam's flow) is derived analytically. The cluster has the profile of a limit cycle dynamics. This shows the ''delay of the changing motion'' of each car, which means the balance of car's moving into and out of a cluster, and this microscopic balance results in the self-organization of a cluster in the diffusion system.

We thank K. Nagel and M. Schreckenberg for suggesting to us the idea of this work through discussion with Y.S. We should comment that T. Nagatani has worked with another simple cellular automaton model based on the OV model [9].

[1] K. Nagel and M. Schreckenberg, J. Phys. I (France) **2**, 2221 (1992); M. Schreckenberg, A. Schadschneider, K. Nagel, and N. Ito, Phys. Rev. E **51**, 2939 (1995).
[2] T. Nagatani, J. Phys. Soc. Jpn. **62**, 3837 (1993); Phys. Rev. E **51**, 922 (1995).
[3] B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, 2335 (1993); **50**, 54 (1994).
[4] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama, Phys. Rev. E **51**, 1035 (1995); Jpn. J. Ind. Appl. Math. **11**, 203 (1994).
[5] M. Bando, K. Hasebe, A. Nakayama, K. Nakanishi, A. Shibata, and Y. Sugiyama, J. Phys. I (France) **5**, 1389 (1995).
[6] T. S. Komotsu and S.-i. Sasa, Phys. Rev. E **52**, 5574 (1995).
[7] D. Helbing, Complex Syst. **6**, 391 (1992).
[8] *Traffic and Granular Flow* edited by D. E. Wolf, M. Schreckenberg, and A. Bachem (World Scientific, Singapore, 1996).
[9] T. Nagatani, Physica A **233**, 137 (1996).